METHOD OF DIMENSIONLESS COEFFICIENTS FOR ANALYSIS OF STRUCTURALLY ORTHOTROPIC PLANE STRUCTURES

PART 1

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1. INTRODUCTION

A rapid and sufficiently exact method of analysis of structurally orthotropic plane structures – such as e.g. unequally reinforced slabs, slabs stiffened by beams, grids with or without slabs – will always be of great interest to a specialist. In the present paper the fundaments of a theory, called the method of dimensionless coefficientst as well as procedures of practical analysis of the bridge systems mentioned above are given. This method enables not only to effectively employ computers in every given case but also to calculate in advance and to tabulate certain dimensionless coefficients (four basic and ten complementary ones) for a rapid and easy analysis of all necessary inner forces in the structure in dependence on three dimensionless parameters: the parameters of lateral stiffness, torsional rigidity and contraction ability. If the present method is applied, much time may be spared for the designers. Moreover, since the transverse contraction is taken into account, the method gives a better picture of the behaviour of structures considered than other methods usually used.

2. BASIC RELATIONS

2.1. Materially orthotropic plate

The adoption of usual simplifying assumptions of the technical small-deflection theory of plates, i.e.

- plate material is perfectly elastic and homogeneous
- plate thickness is uniform and small in comparison to the other dimensions of the plate

- the Kirchhoff-Love's hypothesis holds true, i.e. linear fibres which are initially normal to the middle surface of the plate remain straight and normal to the middle surface of the plate after bending
- normal stresses in the direction transverse to the plane of the plate are negligible and the thickness of the plate does not undergo any deformation during bending
- plate deformations are small in comparison with the plate thickness
- there is no normal strain in planes tangent to the middle surface of the plate
- the body forces are either be disregarded or assumed to be a part of external load
- loading vector is perpendicular to the plane of the plate,

results in the well – known partial differential equation of anisotropic plate

$$(1) \quad \varrho_{11} \frac{\partial^4 w}{\partial x^4} + 4\varrho_{14} \frac{\partial^4 w}{\partial x^3 \partial y} + 2(\varrho_{12} + 2\varrho_{44}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4\varrho_{24} \frac{\partial^4 w}{\partial x \partial y^3} + \varrho_{22} \frac{\partial^4 w}{\partial y^4} - p(x, y) = 0$$

provided that the conventional procedure of the plate theory has been anployed.

Introducing in agreement with [5] the reciprocal theorem of Betti

$$(2) \quad \nu_x \varrho_y = \nu_y \varrho_x$$

or

$$\nu_x E_y = \nu_y E_x$$

respectively, Eq. (1) reverts in the case of an orthotropic plate to

$$(3) \quad \varrho_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + \varrho_y \frac{\partial^4 w}{\partial y^4} - p(x, y) = 0\,,$$

where

$$(4) \quad 2H = 4\gamma_{xy} + \varrho_x \nu_y + \varrho_y \nu_x$$

and ϱ_x, ϱ_y, γ_{xy} denote flexural and torsional rigidities of the plate.[1])

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[1]) For a materially orthotropic plate we have

$$\varrho_x = \frac{E_x d^3}{12(1 - \nu_x \nu_y)}\,, \quad \varrho_y = \frac{E_y d^3}{12(1 - \nu_x \nu_y)}\,, \quad \gamma_{xy} = \frac{G d^3}{12}$$

Introducing [5, 9]

$$G = \frac{\sqrt{(E_x E_y)}}{2(1 + \sqrt{(\nu_x \nu_y)})}\,,$$

with some calculation we obtain

$$2H = 2\sqrt{(\varrho_x \varrho_y)}\,.$$

Since

$$2H = 2\varrho = \frac{E d^3}{6(1 - \nu^2)}$$

for an isotropic plate, Eq. (3) reduces to

$$\nabla^4 w = \frac{1}{\varrho} p(x, y)$$

in this case.

2.2. Structurally orthotropic plane structures

In the foregoing section the material has been assumed to be homogeneous and orthotropic, i.e. its elastic properties were symmetric with respect to three orthogonal planes. In practice, the plate orthotropy appears either because of an unequal reinforcement of prestress of the plate in two perpendicular directions, or because of the connection of the plate to a set of beams or girders either in longitudinal or in transversal direction or in the both directions, respectively. Similar result can be reached by prevented or by intentional reduction of some force transfer in the transverse direction (multi-beam bridges).

Thus, a materially orthotropic plate seems to be the first limit case while in the second limit case the structure is represented by only two sets of parallel beams providing generally a skew crossing. However, in most cases of practical interest the both sets are perpendicular to each other with the girders (called also main – or longitudinal beams) in the span direction and lateral (cross – or transversal –) beams in the transverse direction. Very often the bridge deck or a floor slab is rigidly fixed to one or to both the beam set. The beams are from reinforced concrete, prestressed concrete or steel (mostly of closed box-section with new constructions); since rigid connections are made at the points of intersection, the grid elements always resist in torsion. The slab is mostly made from reinforced or prestressed concrete regardless of the beam material. Because of the comparatively great thickness of the slab it is impossible to neglect its composite action in bending, torsion and shear. Relative importance of those two elements – the system of beams and the plate – varies with varying structural arrangement. There is a continuous series of structures starting with the grid with a very thin plate (or even without it) over a grid with a thick plate up to a true plate of constant thickness. Similarly, the numbers of girders and cross beams may vary in wide limits. The transition from a simple grid to an orthotropic plate is accompanied by a considerable change in the influence of torsion as well as the transverse contraction.

Many times it has been shown that the behaviour of all those structures may be described – performing a transtition to an equivalent plate – by a partial differential equation analogous to Eq. (3), i.e. the equation

$$\varrho_T \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + \varrho_P \frac{\partial^4 w}{\partial x^4} = p(x, y) . \tag{5}$$

The transition from a material to a structural orthotropy is formally defined by the change in the indices of the coefficients standing at the first and third members in Eq. (5) and denoting the unit flexural rigidities (i.e. flexural rigidities of the actual structure per unit length in the longitudinal and transverse directions respectively, with respect to the centroidal axes of the cross-section). The torsional rigidity $4\gamma_{xy}$ of an orthotropic plate at the middle member in the equation is replaced by the sum of unit torsional rigidities γ_T and γ_P of the structure in both orthotropy directions (i.e. torsional rigidities of the actual structure per unit length) that is

$$4\gamma_{xy} = \gamma_T + \gamma_P \,. \tag{6}$$

The torsional rigidity of a substitute structurally orthotropic plate varies between zero and the plate (full) value.[1])

The influence of transverse contraction is marked in a general grid-work system to a greater extent only with the proper plate (represented by the floor or the deck) or with grid-work systems consisting of very fine net of crossing prismatic bars. If the continuity of the cross-section is interrupted in some horizontal planes, the transverse distribution of deformation is prevented. Hence, the influence of transverse contraction is always lower with a substitute structurally orthotropic plate than with a full plate from the same material.

A similar relation to Eq. (2), i.e. the equation

$$\nu_T \varrho_P = \nu_P \varrho_T \tag{7}$$

will be adopted even for structurally orthotropic plates in what follows where, however, ν_T, ν_P are no more the Poisson's ratios in the true sense. These quantities denote the influence of the stress $\sigma_y(\sigma_x)$ upon the deformation $\varepsilon_x(\varepsilon_y)$ and conversely due, however, not to the anisotropy of material but to the structural orthotropy. Naturally, Eq. (7) does not hold exactly with structurally orthotropic plates; it is satisfied in the special case of affine orthotropy [5] for which

$$2H = 2\sqrt{(\varrho_T \varrho_P)} \,.$$

A more detailed analysis of the problem has shown [3] that the errors due to the use of the relation (7) for structurally orthotropic plates are sufficiently small in cases of practical interest.

The coefficient at the middle member in Eq. (5) can be – in agreement with Eq. (4) – written as follows

$$2H = (\varrho_T \nu_P + \varrho_P \nu_T) + (\gamma_T + \gamma_P)$$

or making use of Eq. (7)

$$2H = 2\varrho_P \nu_T + (\gamma_T + \gamma_P) \,. \tag{8}$$

The value of this coefficient depends not only upon the torsional rigidities of the structure in both directions but also upon its contraction ability.

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[1]) One can hardly imagine a substitue structurally orthotropic plate having the torsional rigidity greater than an actual (materially) orthotropic or isotropic plate.

Bending moments in a structurally orthotropic plate are defined by the formulae

(9)
$$M_T = -\varrho_T \left(\frac{\partial^2 w}{\partial x^2} + \nu_P \frac{\partial^2 w}{\partial y^2} \right)$$
$$M_P = -\varrho_P \left(\frac{\partial^2 w}{\partial y^2} + \nu_T \frac{\partial^2 w}{\partial x^2} \right)$$

while for the twisting moments we have

(10)
$$M_{TP} = \gamma_T \frac{\partial^2 w}{\partial x \partial y}$$
$$M_{PT} = -\gamma_P \frac{\partial^2 w}{\partial x \partial y}$$

which means that for $\gamma_T \neq \gamma_P$ generally $|M_{xy}| \neq |M_{yx}|$ in an equivalent structurally orthotropic plate.

For shearing forces we have

(11)
$$Q_T = -\varrho_T \frac{\partial^3 w}{\partial x^3} - (\varrho_T \nu_P + \gamma_P) \frac{\partial^3 w}{\partial x \partial y^2}$$
$$Q_P = -\varrho_P \frac{\partial^3 w}{\partial y^3} - (\varrho_P \nu_T + \gamma_T) \frac{\partial^3 w}{\partial x^2 \partial y}$$

and the reactions are given by the expressions

(12)
$$\bar{Q}_T = -\varrho_T \frac{\partial^3 w}{\partial x^3} - (\varrho_T \nu_P + \gamma_T + \gamma_P) \frac{\partial^3 w}{\partial x \partial y^2}$$
$$\bar{Q}_P = -\varrho_P \frac{\partial^3 w}{\partial y^3} - (\varrho_P \nu_T + \gamma_T + \gamma_P) \frac{\partial^3 w}{\partial x^2 \partial y} \, .$$

The evaluation of unit flexural rigidities ϱ_T, ϱ_P and of the torsional reigidities γ_T, γ_P is carried out for the actual shape of the construction; moreover it is necessary to take accound of the transverse contraction influence upon individual parts of the cross-section and their mutual interaction [2].

2.3. Limit cases

Structural orthotropy is characterized by unequal rigidities in both the orthotropy directions. If the plate thickness is constant and if the cross-section (i.e. the moment of inertia) does not vary unequal rigidities must be due to the different material characteristics only. Consequently, the material orthotropy seems to be the first limit case of structural orthotropy.

It can be proved [4] that a simple grid (or two systems of parallel prismatic bars or fibres) is the second limit case of a structurally orthotropic plane structure. If the prismatic elements are rigidly connected at their points of intersection and resist in torsion, the behaviour of the system considered is fully described by Eq. (5). If both the systems of prismatic elements are not rigid in torsion (the so called "non-torsion grid") — or if they are hinged at the crossings — Eq. (5) reduces to the simple form

$$\varrho_T \frac{\partial^4 w}{\partial x^4} + 2\varrho_P \nu_T \frac{\partial^4 w}{\partial x^2 \partial y^2} + \varrho_P \frac{\partial^4 w}{\partial y^4} - p(x, y) = 0 . \tag{5a}$$

The influence of transverse contraction may be neglected, provided that the spacing of prismatic elements is sufficiently large; then Eq. (5a) reads simply

$$\varrho_T \frac{\partial^4 w}{\partial x^4} + \varrho_P \frac{\partial^4 w}{\partial y^4} - p(x, y) = 0 . \tag{5b}$$

2.4. Boundary conditions

A simple bridge-type construction is assumed, i.e. a construction simply supported along the two apposite edges $x = 0$ and $x = l$, and free along the remaining edges $y = \pm b$ (Fig. 1). If use is made of the Kirchhoff's forces [10] at $y = \pm b$, the boundary conditions are as follow:

$$w_{(x=0;l)} = 0$$

$$\left(\frac{\partial^2 w}{\partial x^2} + \nu_P \frac{\partial^2 w}{\partial y^2}\right)_{(x=0;l)} = 0 \tag{13}$$

or with respect to the fact

$$\left(\frac{\partial^2 w}{\partial y^2}\right)_{(x=0;l)} = 0$$

simply

$$\left(\frac{\partial^2 w}{\partial x^2}\right)_{(x=0;l)} = 0 \tag{14}$$

$$\left(\frac{\partial^2 w}{\partial y^2} + \nu_T \frac{\partial^2 w}{\partial x^2}\right)_{(y=\pm b)} = 0 \tag{15}$$

and

$$\left(\varrho_P \frac{\partial^3 w}{\partial y^3} + [\nu_T \varrho_P + \gamma_T + \gamma_P] \frac{\partial^3 w}{\partial x^2 \partial y}\right)_{(y=\pm b)} = 0 . \tag{16}$$

Inspection of the basic equation (5) and of the boundary conditions (13) – (16) shows that all the cross-sectional and material properties of the structure may be expressed in terms of three dimensionless parameters.

1. In agreement with [6] the relative lateral flexibility of the structure (the equivalent plate) is given by the dimensionles parameter

$$\vartheta = \frac{b}{l}\sqrt[4]{\frac{\varrho_T}{\varrho_P}} \tag{17}$$

which will be referred to as the *parameter of lateral stiffness* of the construction in what follows. The greater ϑ, the more flexible is the lateral stiffness. The parameter ϑ involves not only the cross-sectional charakteristics but also the in-plane dimensions of the plate. The determination of this parameter its usually not difficult. The influence of the material appears only in the case that the material exhibits different properties in both the orthotropy directions. If the structure has an infinite rigidity in the transverse direction, the parametr of lateral stiffness vanishes. With a structure which is perfectly non-rigid in the transverse direction, this parameter will tend to infinity. However, practical limits for this parameter in current cases of structures are 0·05 ÷ ÷ 5·0.

Consequently, the theoretical limits are

$$0 \leqq \vartheta \leqq \infty$$

while the practical limits become

$$0{\cdot}05 \leqq \vartheta \leqq 5{\cdot}0\,.$$

2. Keeping in mind that the actual structure is replaced by an equivalent plate, the flexural rigidity is given by the relation

$$\varrho_T = \frac{\varrho'_T}{1 - \nu_T\nu_P} \tag{18}$$

where ϱ_T denotes the unit flexural rigidity involving the transverse contraction influence, and ϱ'_T means the same rigidity without the contraction influence.

Substituting for ν_P from Eq. (7) into the above equation, we obtain

$$\varrho_T = \frac{\varrho'_T}{1 - \nu_T^2 \dfrac{\varrho_P}{\varrho_T}} = \frac{\varrho'_T}{1 - \eta^2} \tag{19}$$

where *the parametr contraction ability* of the structure has been introduced defined by the expresion

$$\eta = \nu_T \sqrt{\frac{\varrho_P}{\varrho_T}}\,. \tag{20}$$

From Eq. (19) it follows that

(21) $$\eta = \sqrt{\frac{\varrho_T - \varrho'_T}{\varrho_T}}$$

The parameter of contraction ability depends partly upon the flexural rigidities ratio, and partly upon the transverse contraction coefficient ν_T. The smaller the transversal rigidity of the structure (i.e. the greater ϑ), the smaller is the influence of transverse contraction. The greatest value of this parameter is ν_T (for $\varrho_P/\varrho_T = 1$ since $\varrho_P/\varrho_T > 1$ has no practical application); while the second extremum, which is zero, is reached for negligible transverse flexural rigidity.

Consequently,

$$0 \leqq \eta \leqq \nu_T$$

for all structures of practical interest.

3. The multiplier of the middle member with respect to Eq. (8) is

$$2H = 2\varrho_P \nu_T + (\gamma_T + \gamma_P) .$$

Dividing this equation by $2\sqrt{(\varrho_T \varrho_P)}$ and employing Eq. (20), we obtain in

(22) $$\frac{2H}{2\sqrt{(\varrho_T \varrho_P)}} = \eta + \frac{\gamma_T + \gamma_P}{2\sqrt{(\varrho_T \varrho_P)}}$$

the non-dimensional coefficient of the middle member in the Huber's equation of the equivalent orthotropic plate. Relative torsional rigidity of the structure is, according to [11], given by the dimensionless *torsional parameter*

(23) $$\alpha = \frac{\gamma'_T + \gamma'_P}{2\sqrt{(\varrho'_T \varrho'_P)}}$$

where ϱ'_T, ϱ'_P, γ'_T, γ'_P denote unit flexural and torsional rigidities of the structure for zero transverse contraction coeficient of the material. Assuming

$$\varrho_T = \frac{\varrho'_T}{1 - \eta^2}, \quad \varrho_P = \frac{\varrho'_P}{1 - \eta^2}, \quad \gamma_T = \frac{\gamma'_T}{1 + \eta}, \quad \gamma_P = \frac{\gamma'_P}{1 + \eta}$$

and making use of Eq. (20), we have

(24) $$\alpha = \frac{\gamma_T + \gamma_P}{2(1 - \eta)\sqrt{(\varrho_T \varrho_P)}} .$$

Combining Eqs. (22) and (24), we obtain

(25) $$2H = 2\varepsilon \sqrt{(\varrho_T \varrho_P)}$$

where

(26) $$\varepsilon = \eta + \alpha(1 - \eta) = \alpha + \eta(1 - \alpha)$$

is the *middle-member parameter of the Huber's equation* of the equivalent orthotropic plate. The torsional parameter of an arbitrary structurally orthotropic system lies evidently between the limits[1])

$$0 \leqq \alpha \leqq 1 .$$

The middle-member parametr of the Huber's equation ε is then bounded by the limits

$$\eta \leqq \varepsilon \leqq 1 .$$

4. METHOD OF DIMENSIONLESS COEFFICIENTS

Many of the current methods of solving the basic differential equation of the considered structures lead to practically untractable calculations, if the designer's equipment represent only a slide rule or a calculating machine. There are thus two ways of attacking the problem: either to employ the current methods in connection with a computer, or, which seems to be more simple, to make use of a practically applicable approximate method. One of the latter methods is the method of dimensionless coefficients; by means of those coefficients all the inner force components are expressed which are necessary for the design of the structure. This method is based on two fundamental assumptions:

a) It is possible to investigate an equivalent orthotropic plate instead of the actual system represented either by an orthogonally stiffened plate or by a grid (either connected with the plate or without the plate). The rigidities of the actual system are continuously distributed so that the elastic properties per unit legth of the actual and equivalent systems are the same. The equivalent plate is then solved as a plane problem. If the values of dimensionless coefficients are tabulated, the calculation simplifies to a great extent, and at the same time the accuracy of the results shows to be quite satisfactory even in the case that all the remaining numerical operations are performed on a slide rule.

b) Actual loading corresponding to the initial system is replaced by a loading expanded into a sine-series in the girder direction

$$p(x) = \sum_m p_m \sin \frac{m\pi x}{l}$$

where p_m denotes the amplitude of the m-th member of the appropriate Fourier series (Fig. 2). In the transverse direction the loading distribution of the equivalent

[1]) Since any structurally orthotropic plane structure represented by a plate and by prismatic elements may be interpreted as a plate, with thickness equal to the greatest depth of the cross-section of the structure where, however, the mass between adjacent prismatical elements have been taken away. The upper limit (the unity) can be reached only in the case of a cross-system of girders having small spacings. In this very case the structure represents actually a plate with small openings.

plate is defined by the basic equation of the problem, Eq. (5). Massonnet [11] has shown that in most cases the assumption of the loading distribution in the form of the first member

$$p(x) = p_1 \sin \frac{\pi x}{l}$$

leads to sufficiently accurate results.

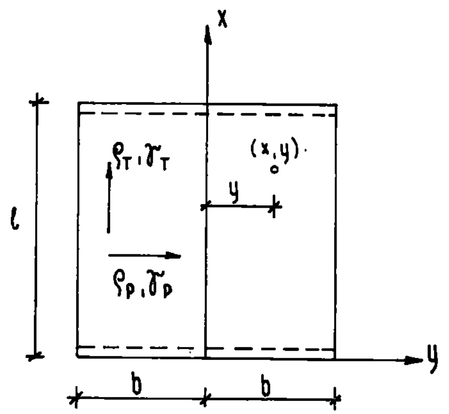


Fig. 1.

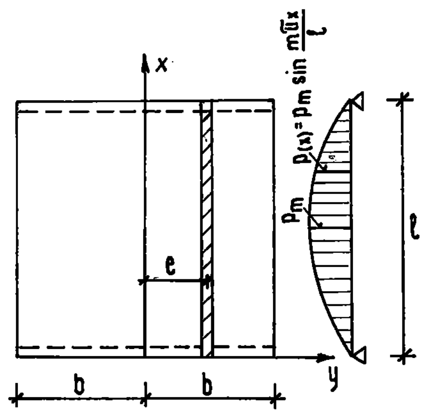


Fig. 2.

The above given two basic assumptions influence the distribution of loading effects in the transverse direction only. The rest of the analysis is governed by the usual rules of structural mechanics.

The solution of the basic equation (5) is obtained as a sum of the particular integral ${}^1w$ and of the integral ${}^2w$ of the homogeneous equation, that is

$$w(x, y) = {}^1w + {}^2w \,. \tag{27}$$

The deflection surface $w(x, y)$ of a bridge type plate under a harmonic loading

$$p(x) = \sum_m p_m \sin \frac{m\pi x}{l} \tag{28}$$

is represented by a similar function

$$w(x, y) = \sum_m W(y)_m \sin \frac{m\pi x}{l} \tag{29}$$

which in the X-direction is governed by the same harmonic law. The values of $W(y)$ are then found from Eq. (27) at the plate edges.

All the inner force components, expressed by derivatives of the function $W(x, y)$, as given by (29), are then put down as a product of two functions, one of which is a function of the dimensionless parameters ϑ, α, η (resp. ν), φ (defining the relative – non-dimensional – transversal coordinate of the point in which the effect considered is to be determined), Ψ (defining the relative transversal coordinate of the point where the loading is acting), and the second of them having the dimension of the respective inner force components and depending upon the outer dimensions of the structure, the magnitude of the loading, flexural and torsional rigidities, and upon the non-dimensional coordinate $\xi = x/l$ (defining the longitudinal coordinate of the point in which the effect considered is to be determined). This fact simplifies the calculation to a great extent, since it is possible to tabulate the first function, which is rather complicated in form, in dependence upon the dimensionless parameters. The calculation of the second function as well as the calculations of the respective inner force components itself is then quite elementary – the extent of the work necessary is about the same as when solving a truss.

From the fact that the first function is non-dimensional, the name dimensionless coefficient has been derived. Nevertheless, the method is practically applicable only owing to the following two facts:

A. A general line loading expanded into a Fourier series in the X-direction requires usually more than one member of the series for sufficiently accurate expression of the inner forces (moments, shearing forces, reactions). If the coefficients must be evaluated for more members of the series, the calculations become considerably complicated and the tables would be of no use because of their great number. Fortunately, a more detailed analysis of the derived equations shows that the m-th member of the series corresponding to the loading distribution $p(x) = \sum p_m \sin m\pi x/l$ upon a system whose lateral stiffness is ϑ is equal to the first member upon a structural system with an m-times more flexible stiffness in transverse direction, i.e. with the parameter of lateral stiffness equal to $m\vartheta$. In other words: the lateral stiffness for the loading $p(x)_m = p_m \sin m\pi x/l$ becomes m-times more flexible than for the loading $p(x)_1 = p_1 \sin \pi x/l$. This knowledge is very important since it enables to tabulate the dimensionless coefficients for the first member only, that is for $m = 1$, but to make use of them for any arbitrary member of the expansion. The coefficients $X(y)_1, X(y)_2, X(y)_3 \dots X(y)_m$ for the loading $p_1 \sin \pi x/l$, $p_2 \sin 2\pi x/l$, $p_3 \sin 3\pi x/l \dots$ $p_m \sin m\pi x/l$ are obtained from the tables of the values $X(y)_1$ successively for ϑ, 2ϑ, $3\vartheta \dots m\vartheta$.

B. Dimensionless coefficients X depend aside from φ, ψ also on the dimensionless parameters ϑ, α, η which may assume arbitrary values within their intervals of definition (cf. Sec. 3). Thus, it would be necessary to tabulate each coefficient X in each system φ, ψ for all the combinations of the three parameters ϑ, α, η employing sufficiently fine subdivision. This fact would prevent practical application of the method described. The possibility of application is, however, saved by the fact that the variation of the dimensionless coefficient X with α and η, respectively, in their definition intervals (between the limits) is represented by a smooth, mostly monotonic and easily expressible interpolation function. It has been shown that the following parabolic interpolation formula

$$X_k = X_{\min} + (X_{\max} - X_{\min})\, F(k) \tag{30}$$

may be employed for the interpolation between the limit (and basic) values of the parameters $\alpha = 0$, $\alpha = 1$ and $\eta = 0$, $\eta = 0{\cdot}25$, respectively.

In this way the necessary tables of dimensionless coefficients are reduced to an reasonable extent: as concerns the practically important values of the first parameter ϑ (from 0·05 up to 5·0) the coefficients must be tabulated only for the two limit values of the second and third parameters, respectively ($\alpha = 0$, $\alpha = 1$ and $\eta = 0$, $\eta = 0{\cdot}25$).

Analysis of the form of the expressions for the dimensionless parameters as functions of α and η has shown that it is more convenient to interpolate first the value of $X = f(\alpha, \eta)$ with respect to η (successively for $\alpha = 0$ and $\alpha = 1$) and then with respect to α. Thus, first the functions $X_{0;\eta}$, and $X_{1;\eta}$ are expressed in terms of the basic functions $X_{0;0}$, $X_{0;0,25}$, $X_{1;0}$, $X_{1;0,25}$. After that the interpolation between $X_{0;\eta}$ and $X_{1;\eta}$ is performed, that is the function $X_{\alpha,\eta}$ is found. The value of $F(k)$ differs slightly for different dimensionless parameters; this value depends not only upon α, η but also upon ϑ, φ and ψ which means that it is necessary to determine $F(k)$ for each dimensionless coefficient separately.

4.1. Homogeneous equation solution of a structurally orthotropic plane structure

The homogeneous form of Eq. (5) is obtained for vanishing right-hand member. The solution of such an equation is assumed according to M. Lévy in the form of a series

$$^{2}w_{(x,y)} = \sum_{m=1}^{\infty} Y(y)_m \sin \frac{m\pi x}{l} \tag{31}$$

each member of which satisfies the boundary conditions (13) along the two simply-supported edges $x = 0$ and $x = l$, respectively. The function $Y(y)_m$ is then found from the condition that the homogeneous differential equation corresponding to Eq. (5) be satisfied. In a manner similar to that employed in [11] the following function is arrived at

$$Y_m = e^{mny}[A_m \cos mty + \bar{B}_m \sin mty] + e^{-mny}[C_m \cos mty + \bar{D}_m \sin mty] \tag{32}$$

where

$$n = \omega \sqrt{\left(\frac{1+\varepsilon}{2}\right)}, \quad t = \omega \sqrt{\left(\frac{1-\varepsilon}{2}\right)}, \quad \omega = \frac{\pi}{l} \sqrt[4]{\left(\frac{\varrho_T}{\varrho_P}\right)} \tag{33}$$

$$\bar{B}_m = \frac{B_m}{\sqrt{\left(\frac{1-\varepsilon}{2}\right)}}, \quad \bar{D}_m = \frac{D_m}{\sqrt{\left(\frac{1-\varepsilon}{2}\right)}}.$$

4.2. Particular solution of a structurally orthotropic plane structure

In order to find a particular solution to our problem, it is convenient to consider a simply-supported plate strip (i.e. a plate of infinite width simply-supported along $x = 0$ and $x = l$), instead of the given bridge plate. The form of the particular solution depends upon the form of the loading. General loading may usualy be expressed as a sum of a uniformly distributed loading (across the width) and a line loading. The loading distribution in the X-direction is quite immaterial since it can be approximated by a Fourier series – Eq. (28). Thus, only the two basic special cases of loading described above will be considered in what follows.

4.2.1. Line loading harmonically distributed in the X-direction

A plate strip in the Y-direction under a line loading harmonically distributed in the X-direction is considered

$$p(x)_m = p_m \sin \frac{m\pi x}{l}$$

(Fig. 3).

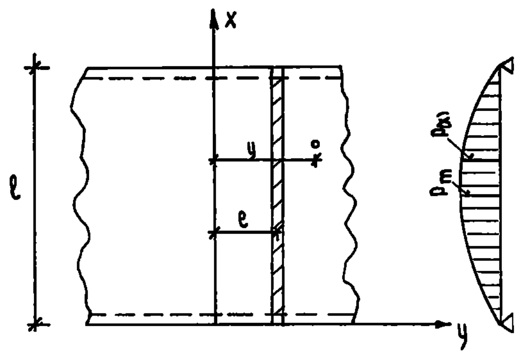


Fig. 3.

In a similar manner as used in [11] taking into account the fact that the twisting moment influence upon the shearing force at the point of application of the loading vanishes in the case considered, which means that the condition of the shearing force distribution in the X-direction

$$-\varrho_P \left[\frac{\partial^3 w}{\partial y^3} + \left(\frac{\gamma_T}{\varrho_P} + \nu_T\right)\frac{\partial^3 w}{\partial x^2 \partial y}\right]_{y=0} = -\frac{p_m}{2}\sin\frac{m\pi x}{l}$$

may be reduced to the simple form

$$\left[-\varrho_P \frac{\partial^3 w}{\partial y^3}\right]_{y=0} = -\frac{p_m}{2}\sin\frac{m\pi x}{l}\Big),$$

the following particular solution is obtained

$$(34)\quad {}^1w_m = C_m^* \, e^{-mn|y-e|}\left\{\cos mt|y-e| + \sqrt{\left(\frac{1+\varepsilon}{1-\varepsilon}\right)}\sin mt|y-e|\right\}\sin\frac{m\pi x}{l}.$$

$(y \geqq i)$

4.2.2. *Loading distributed uniformly in the Y-direction and harmonically in the X-direction*

The deflection surface of an infinite simply-supported plate strip under the loading considered (Fig. 4) is represented by a cylindrical surface defined by the beam-equation

$$\frac{d^4\, {}^1w_m^0}{dx^4} = \frac{p_{(x)}^0}{\varrho_T} = \frac{p_m^0}{\varrho_T} \sin \frac{m\pi x}{l}$$

where $p^0(x)$ and p_m^0, respectively, denote the loading and loading amplitude per unit width. Particular solution of the above differential equation reads

$$(35) \qquad {}^1w_m^0 = \frac{p_m^0}{\varrho_T} \frac{l^4}{m^4\pi^4} \sin \frac{m\pi x}{l} .$$

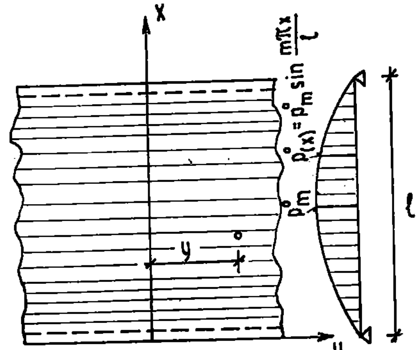


Fig. 4.

4.3. Total solution of a structurally orthotropic plane structure

Total solution of the problem considered is obtained – Eq. (27) – as the sum of the homogeneous equation and particular solutions and depends upon the boundary conditions as well as upon the loading.

4.3.1. *Line loading harmonically distributed in the X-direction*

A simple bridge structure is considered under a line loading acting in the distance e from the X-axis and varying harmonically in the X-direction. Total solution of the present problem is obtained as a sum of the results (31) and (34) with respect to Eq. (27)

$$(36) \qquad w_m = {}^1w_m + {}^2w_m = \Big\{ e^{mny}[A_m \cos mty + \bar{B}_m \sin mty] +$$

$$+ e^{-mny}[Cm \cos mty + \bar{D}m \sin mty] + C_m^* e^{-mn|y-e|} \Big[\cos mt|y - e| +$$

$$+ \sqrt{\left(\frac{1+\varepsilon}{1-\varepsilon}\right)} \sin mt\, |y - e| \Big] \Big\} \sin \frac{m\pi x}{l} = W(y)_m \sin \frac{m\pi x}{l} .$$

Constants of integration A_m, B_m, C_m, D_m are calculated from the condition that the total solution w_m satisfies the boundary conditions along the free edges. These conditions follow from Eqs. (15), (16); with a little rearrangement Eq. (16) is

$$\text{(16a)} \qquad \bar{Q}_P \to \left\{\frac{\partial^3 w}{\partial y^3} - \frac{\omega^2 l^2}{\pi^2}\left[\eta + 2\alpha(1-\eta)\right]\frac{\partial^3 w}{\partial x^2 \partial y}\right\}_{y=\pm b} = 0 \, .$$

Carrying out the appropriate differentiation of w_m given by Eq. (36) and applying the boundary conditions (15), (16a), the following four algebraic equations are received for the unknown constants A_m, B_m, C_m, D_m:

At $y = +b$ from Eq. (15) it follows that

$$\text{(37a)} \qquad e^{mnb}\{[(\varepsilon-\eta)\,A_m + \sqrt{(1-\varepsilon^2)}\,\bar{B}_m]\cos mtb + [-\sqrt{(1-\varepsilon^2)}\,A_m + (\varepsilon-\eta)\,\bar{B}_m]\sin mtb\} + e^{-mnb}\{[(\varepsilon-\eta)\,C_m - \sqrt{(1-\varepsilon^2)}\,\bar{D}_m]\cos mtb + [\sqrt{(1-\varepsilon^2)}\,C_m + (\varepsilon-\eta)\,\bar{D}_m]\sin mtb\} + C_m^* e^{-mn|b-e|}\left[(1-\eta)\sqrt{\left(\frac{1+\varepsilon}{1-\varepsilon}\right)}\sin mt|b-e| - (1-\eta)\cos mt|b-e|\right] = 0 \, .$$

At $y = -b$ from Eq. (15), we have

$$\text{(37b)} \qquad e^{-mnb}\{[(\varepsilon-\eta)\,A_m + \sqrt{(1-\varepsilon^2)}\,\bar{B}_m]\cos mtb + [\sqrt{(1-\varepsilon^2)}\,A_m + (-\varepsilon+\eta)\,\bar{B}_m]\sin mtb\} + e^{mnb}\{[(\varepsilon-\eta)\,C_m - \sqrt{(1-\varepsilon^2)}\,\bar{D}_m]\cos mtb - [\sqrt{(1-\varepsilon^2)}\,C_m + (\varepsilon-\eta)\,\bar{D}_m]\sin mtb\} + C_m^* e^{-mn(b+e)}\left[(1-\eta)\left(\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\right)\sin mt(b+e) - (1-\eta)\cos mt(b+e)\right] = 0 \, .$$

At $y = +b$ from Eq. (16a) it follows that

$$\text{(38a)} \qquad e^{mnb}\left\{\left[(-1+\eta)\sqrt{\left(\frac{1+\varepsilon}{1-\varepsilon}\right)}A_m + (1+\eta)\,\bar{B}_m\right]\cos mtb + \left[-(1+\eta)\,A_m + (-1+\eta)\sqrt{\left(\frac{1+\varepsilon}{1-\varepsilon}\right)}\bar{B}_m\right]\sin mtb\right\} + e^{-mnb}\left\{\left[(1-\eta)\sqrt{\left(\frac{1+\varepsilon}{1-\varepsilon}\right)}C_m + (1+\eta)\,\bar{D}_m\right]\cos mtb + \left[-(1+\eta)\,C_m + (1-\eta)\sqrt{\left(\frac{1+\varepsilon}{1-\varepsilon}\right)}\bar{D}_m\right]\sin mtb\right\} + \frac{2}{1-\varepsilon}C_m^* e^{-mn|b-e|}[(\varepsilon-\eta)\sin mt|b-e| + \sqrt{(1-\varepsilon^2)}\cos mt|b-e|] = 0 \, .$$

Fig. 5.

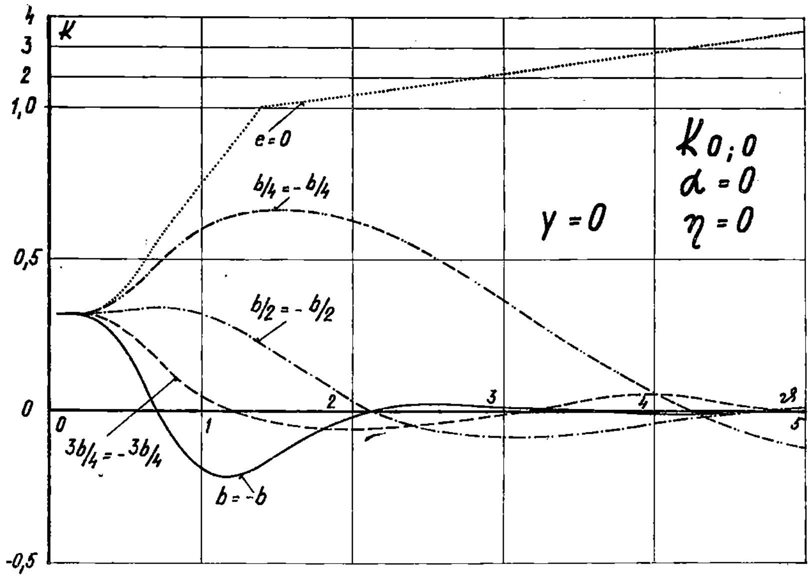
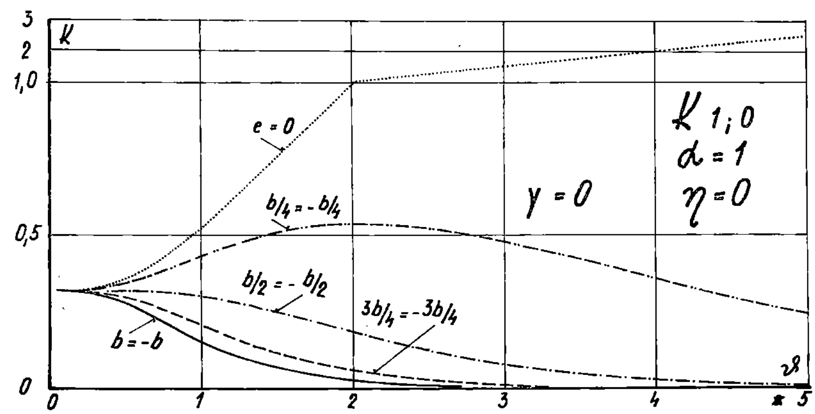


Fig. 6.



And finally at $y = -b$ from Eq. (16a), further

$$
(38b) \quad e^{mnb} \left\{ \left[(1-\eta) \sqrt{\left(\frac{1+\varepsilon}{1-\varepsilon}\right)} C_m + (1+\eta) \bar{D}_m \right] \cos mtb + (1+\eta) C_m - \right.
$$

$$
\left. - (1-\eta) \sqrt{\left(\frac{1+\varepsilon}{1-\varepsilon}\right)} \bar{D}_m \right] \sin mtb \Big\} + e^{-mnb} \left\{ \left[- (1-\eta) \sqrt{\left(\frac{1+\varepsilon}{1-\varepsilon}\right)} A_m + \right. \right.
$$

$$
\left. + (1+\eta) \bar{B}_m \right] \cos mtb + \left[(1-\eta) A_m + (1-\eta) \sqrt{\left(\frac{1+\varepsilon}{1-\varepsilon}\right)} \bar{B}_m \right] \sin mtb \Big\} -
$$

$$
- \frac{2}{1-\varepsilon} C_m^* e^{-mn(b+e)} [(\varepsilon - \eta) \sin mt(b+e) + \sqrt{(1-\varepsilon^2)} \cos mt(b+e)] = 0 .
$$

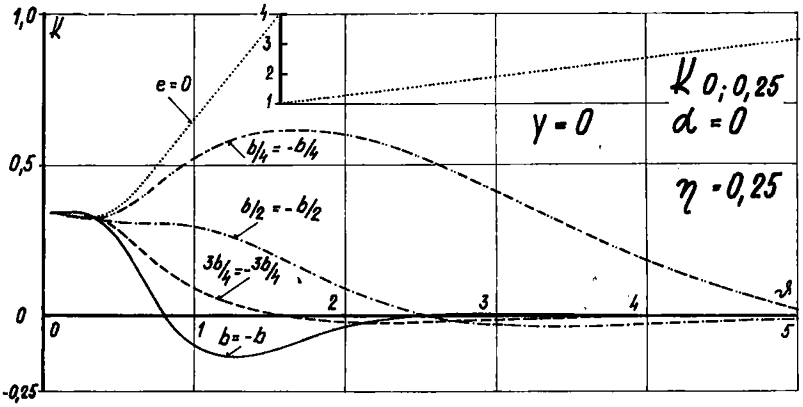


Fig. 7.

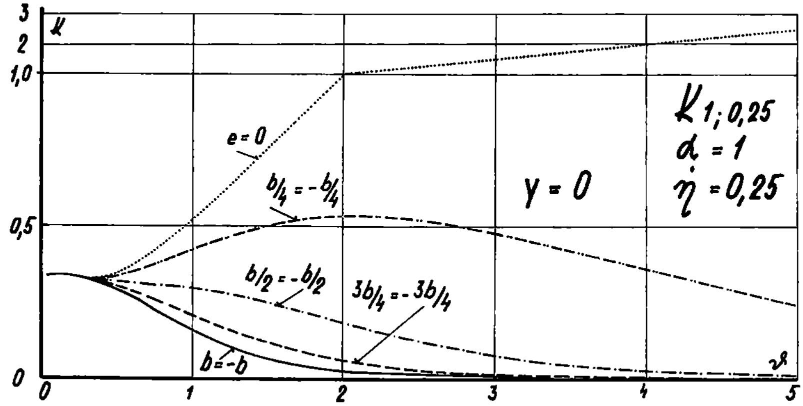


Fig. 8.

Introducing $\psi = \pi e/b$, $\omega = \pi\vartheta/b$, and making use of Eq. (33), we obtain

$$(39) \qquad n(b+e) = \vartheta\sqrt{\left(\frac{1+\varepsilon}{2}\right)}(\pi+\psi) = n'(\pi+\psi)$$

$$t(b+e) = \vartheta\sqrt{\left(\frac{1-\varepsilon}{2}\right)}(\pi+\psi) = t'(\pi+\psi)$$

$$n|b-e| = \vartheta\sqrt{\left(\frac{1+\varepsilon}{2}\right)}(\pi-\psi) = n'(\pi-\psi)$$

$$t|b-e| = \vartheta\sqrt{\left(\frac{1-\varepsilon}{2}\right)}(\pi-\psi) = t'(\pi-\psi)\,.$$

In order to simplify the writing the following notation will now be used

$$(40) \quad K = \left(e^{mn'\pi} + e^{-mn'\pi}\right)\cos mt'\pi$$

$$L = \left(e^{mn'\pi} - e^{-mn'\pi}\right)\cos mt'\pi$$

Fig. 9.

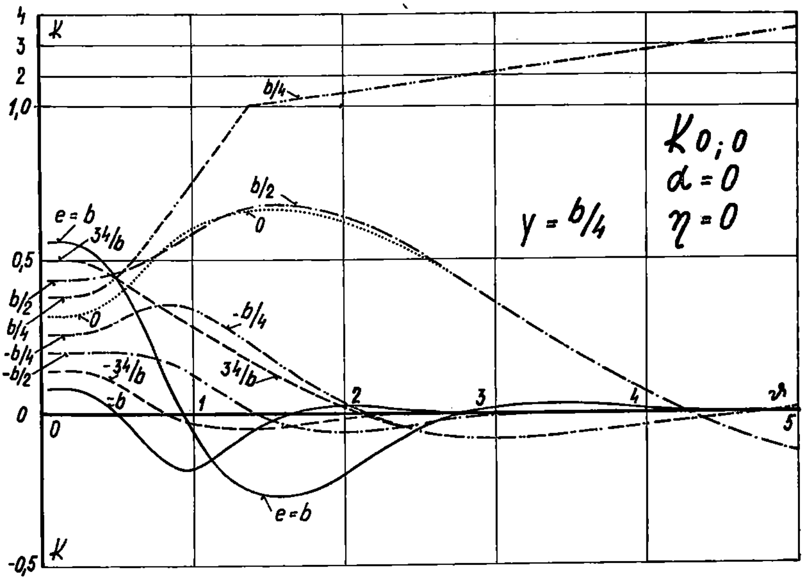
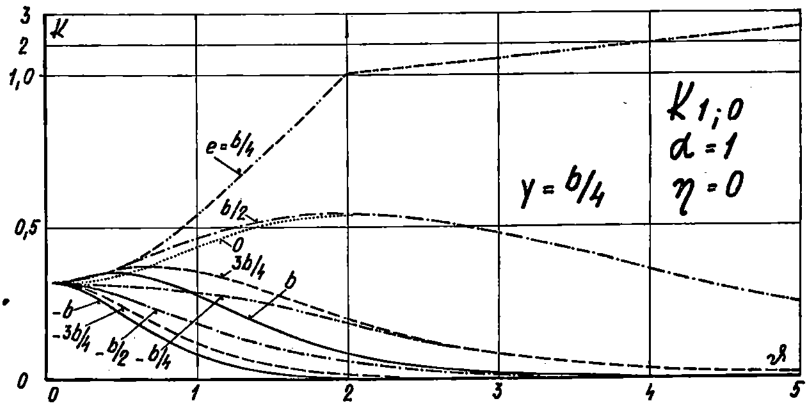


Fig. 10.



$$I = \left(e^{mn'\pi} + e^{-mn'\pi}\right) \sin mt'\pi$$

$$J = \left(e^{mn'\pi} - e^{-mn'\pi}\right) \cos mt'\pi$$

$$E = C_m^* \, e^{-mn'(\pi-\psi)}\left[(1-\eta)\, a \sin mt'(\pi-\psi) - (1+\eta) \cos mt'(\pi-\psi)\right]$$

$$F = C_m^* \, e^{-mn'(\pi+\psi)}\left[(1-\eta)\, a \sin mt'(\pi+\psi) - (1+\eta) \cos mt'(\pi+\psi)\right]$$

$$G = 2C_m^* \, e^{-mn'(\pi-\psi)}\left[\frac{\varepsilon-\eta}{1-\varepsilon} \sin mt'(\pi-\psi) + a \cos mt'(\pi-\psi)\right]$$

$$H = 2C_m^* \, e^{-mn'(\pi+\psi)}\left[\frac{\varepsilon-\eta}{1-\varepsilon} \sin mt'(\pi+\psi) + a \cos mt'(\pi+\psi)\right]$$

$$n_1 = K(\varepsilon-\eta) - J\sqrt{(1-\varepsilon^2)} \qquad n_3 = L(\varepsilon-\eta) - I\sqrt{(1-\varepsilon^2)}$$

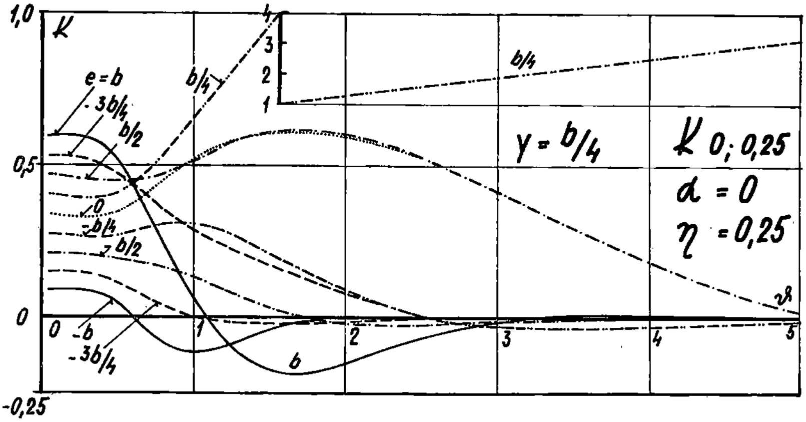


Fig. 11.

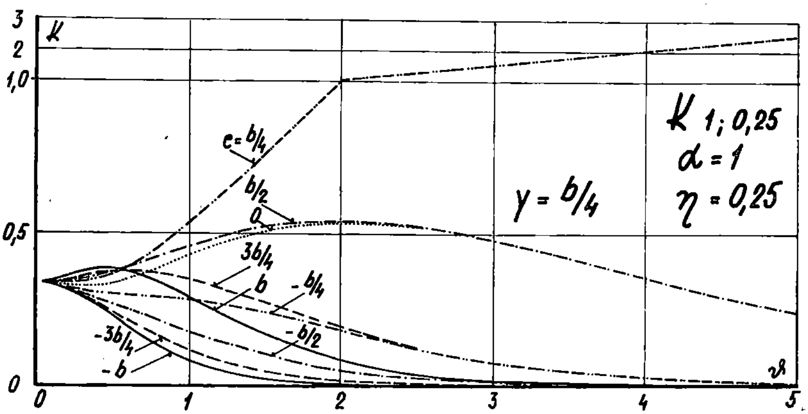


Fig. 12.

$$n_2 = K\sqrt{[2(1+\varepsilon)]} + J\frac{\varepsilon-\eta}{\sqrt{\left(\frac{1-\varepsilon}{2}\right)}} \qquad n_4 = L\sqrt{[2(1+\varepsilon)]} + I\frac{\varepsilon-\eta}{\sqrt{\left(\frac{1-\varepsilon}{2}\right)}}$$

$$a = \sqrt{\left(\frac{1+\varepsilon}{1-\varepsilon}\right)}.$$

Adding and subtracting Eqs. (37a) and (37b), (38a) and (38b), after some calculation we have

$$(41) \qquad \begin{aligned} (A_m + C_m)\, n_1 + (B_m - D_m)\, n_2 + E + F &= 0 \\ (A_m - C_m)\, n_3 + (B_m + D_m)\, n_4 + E - F &= 0 \\ (A_m + C_m)\, n_5 - (B_m - D_m)\, n_6 - G - H &= 0 \\ (A_m - C_m)\, n_7 - (B_m + D_m)\, n_8 - G + H &= 0\,. \end{aligned}$$

Fig. 13.

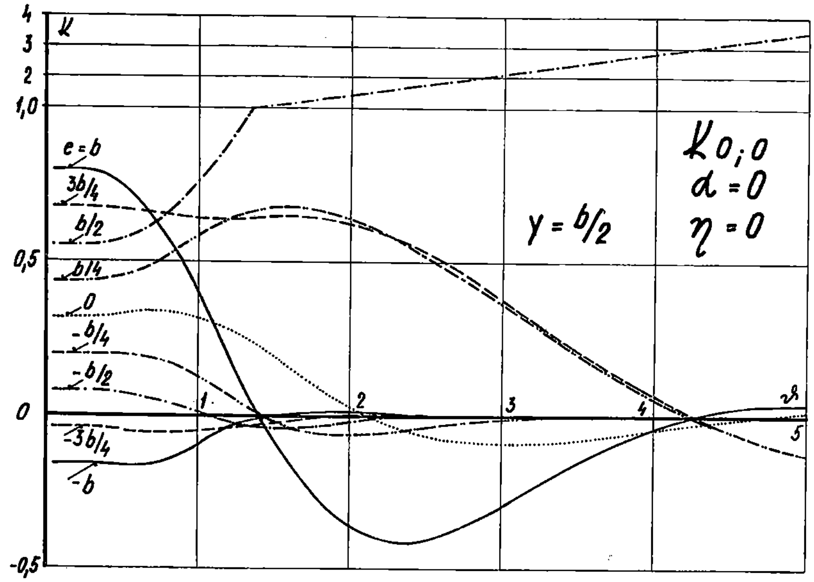
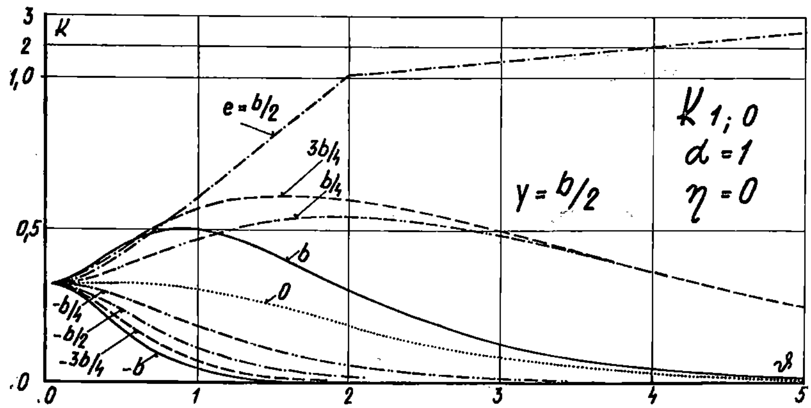


Fig. 14.



Further simplification may be reached by putting

(42)
$$Q = \frac{G - H}{C_m^*} \qquad S = \frac{E + F}{C_m^*}$$

$$R = \frac{G + H}{C_m^*} \qquad T = \frac{E - F}{C_m^*}$$

and further

$$n_1 n_6 + n_2 n_5 = V_1$$

$$n_3 n_8 + n_4 n_7 = V_2$$

$$U = V_1(Qn_4 - Tn_8)$$

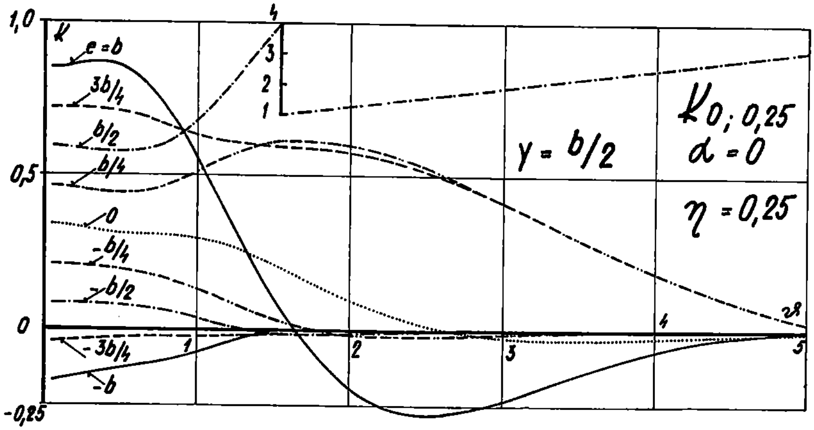


Fig. 15.

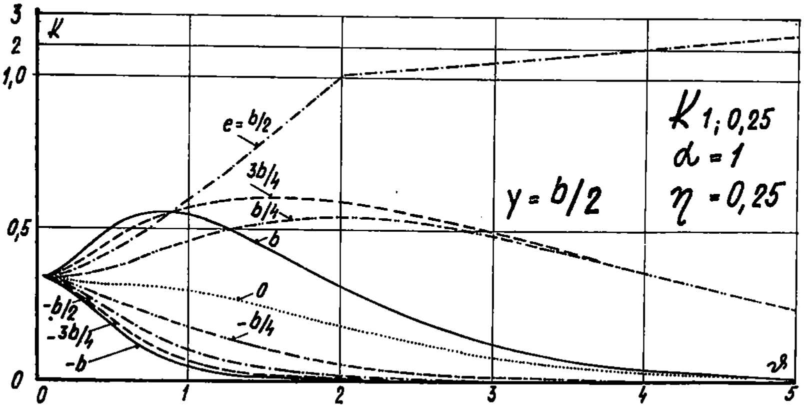


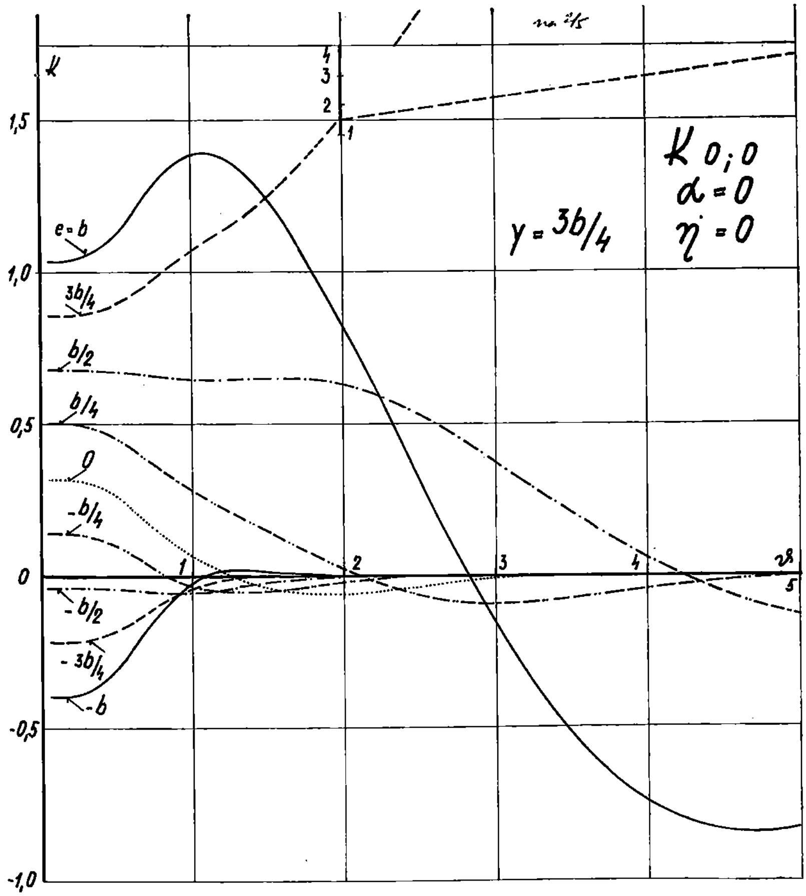
Fig. 16.

$$W = V_2(Rn_2 - Sn_6)$$
$$X = V_2(Rn_1 + Sn_5)$$
$$Y = V_1(Qn_3 + Tn_7)$$

The constants of integration may now be expressed as follows

(43)
$$A_m = \frac{C_m^*}{2V_1V_2}(U + W) = C_m^* A_m'$$
$$B_m = -\frac{C_m^*}{2V_1V_2}(X + Y) = C_m^* B_m'$$
$$C_m = \frac{C_m^*}{2V_1V_2}(W - U) = C_m^* C_m'$$
$$D_m = -\frac{C_m^*}{2V_1V_2}(X - Y) = C_m^* D_m'.$$

Fig. 17.



Substituting the above results into Eq. (36), the total solution of a structurally orthotropic plane structure under a line harmonic loading $p(x) = p_m \sin m\pi x/l$ is obtained. In order to simplify the writing further, we introduce $\varphi = \pi y/b$ and put

$$
\begin{aligned}
(44) \qquad M_{\varphi m} &= e^{mn'\varphi} \cos mt'\varphi \\
N_{\varphi m} &= e^{mn'\varphi} \sin mt'\varphi \\
O_{\varphi m} &= e^{-mn'\varphi} \cos mt'\varphi \\
P_{\varphi m} &= e^{-mn'\varphi} \sin mt'\varphi \\
Q_{|\varphi-\psi|m} &= e^{-mn'|\varphi-\psi|} \cos mt'|\varphi - \psi| \\
P_{|\varphi-\psi|m} &= e^{-mn'|\varphi-\psi|} \sin mt'|\varphi - \psi| \,.
\end{aligned}
$$

The deflection surface is then given by the expression

$$
(36a) \qquad w(x, y)_m = C_m^*\{[A'_m M_{\varphi m} + \bar{B}'_m N_{\varphi m}] + [C'_m O_{\varphi m} + \\
+ \bar{D}'_m P_{\varphi m}] + [O_{|\varphi-\psi|m} + aP_{|\varphi-\psi|m}]\} \sin \frac{m\pi x}{l} \,,
$$

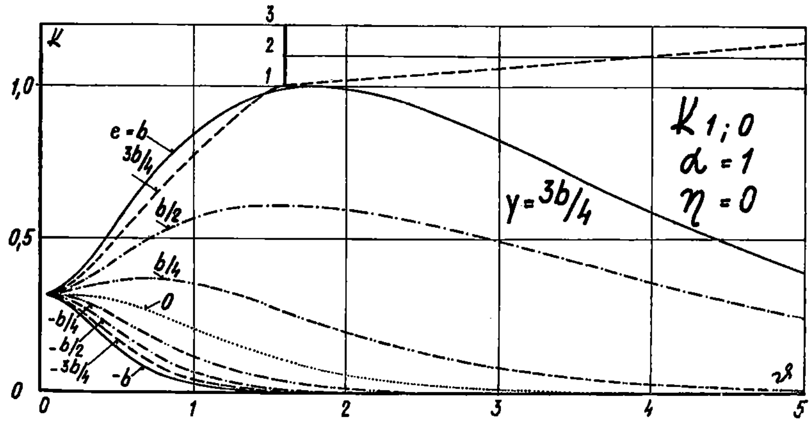


Fig. 18.

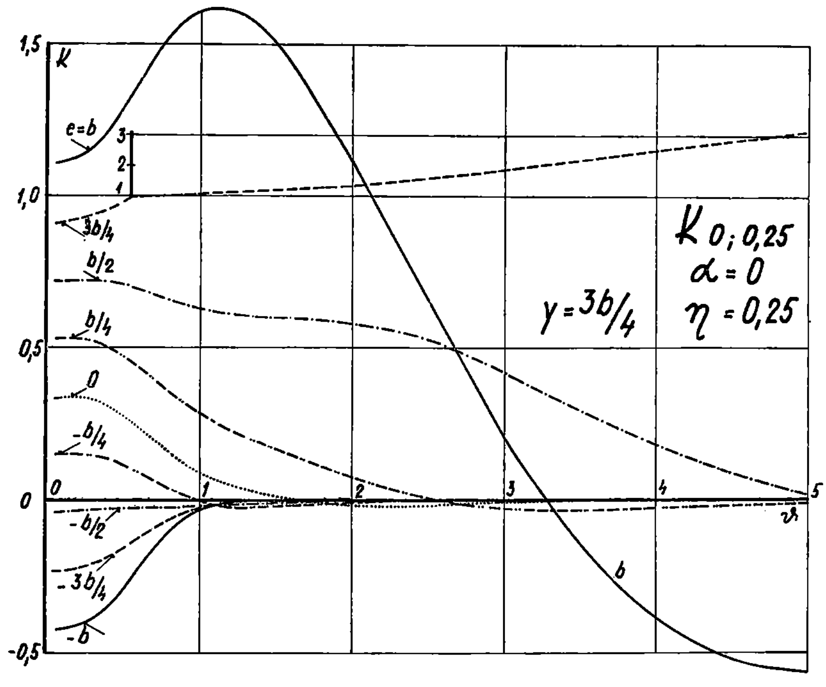


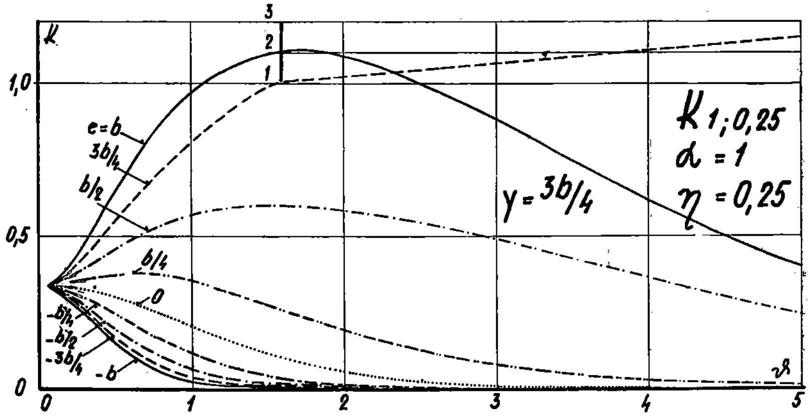
Fig. 19.

or simply

(45) $$w(x, y)_m = \frac{p_m l^4}{2bm^4\pi^3 \varrho_T} K(y)_m \sin \frac{m\pi x}{l}$$

where the ratio standing at the right-hand side has the dimension of the respective quantity (length unity) and $K(y)_m$ denotes *the first dimensionless coefficient* depending upon φ, ψ, ϑ, α and η as follows:

Fig. 20.



(46) $$K(y)_m = \frac{m\vartheta}{\sqrt{[2(1+\varepsilon)]}} \{[A'_m M_{\varphi m} + \bar{B}'_m N_{\varphi m}] + [C'_m O_{\varphi m} - \bar{D}'_m P_{\varphi m}] + [O_{|\varphi-\psi|m} + aP_{|\varphi-\psi|m}]\}\,.$$ [1])

Plots of this first and fundamental dimensionless coefficient $K \equiv K(y)_1$ versus ϑ are given in Figs. 5–24 for $\alpha = 0$, $\alpha = 1$ and $\eta = 0$, $\eta = 0{,}25$ and for various values of φ and ψ, respectively.

4.3.2. Loading distributed uniformly in the width-direction and harmonically in the X-direction

Similarly to the foregoing case, the total solution for the present loading is found as a sum of the homogeneous equation solution (31) and the particular solution (35), that is

(47) $$w^0(x, y)_m = \left\{\frac{l^4 p_m^0}{\varrho_T \pi^4 m^4} + e^{mny}[A_m^0 \cos mty + \bar{B}_m^0 \sin mty] + e^{-mny}[C_m^0 \cos mty + \bar{D}_m^0 \sin mty]\right\} \sin \frac{m\pi x}{l}\,.$$

For the structure of the width 2b a uniformly distributed loading

(48) $$p^0(x) = p_m^0 \sin \frac{m\pi x}{l}$$

[1]) For $\eta = 0$ the present coefficient $K(y)_m$ is the $1/\pi$-multiple of the Massonnet's coefficient [11, 12], i.e.

$$[K(y)_m]_{\eta=0} = \frac{1}{\pi} K^{\mathrm{MASS}}(y)_m\,.$$

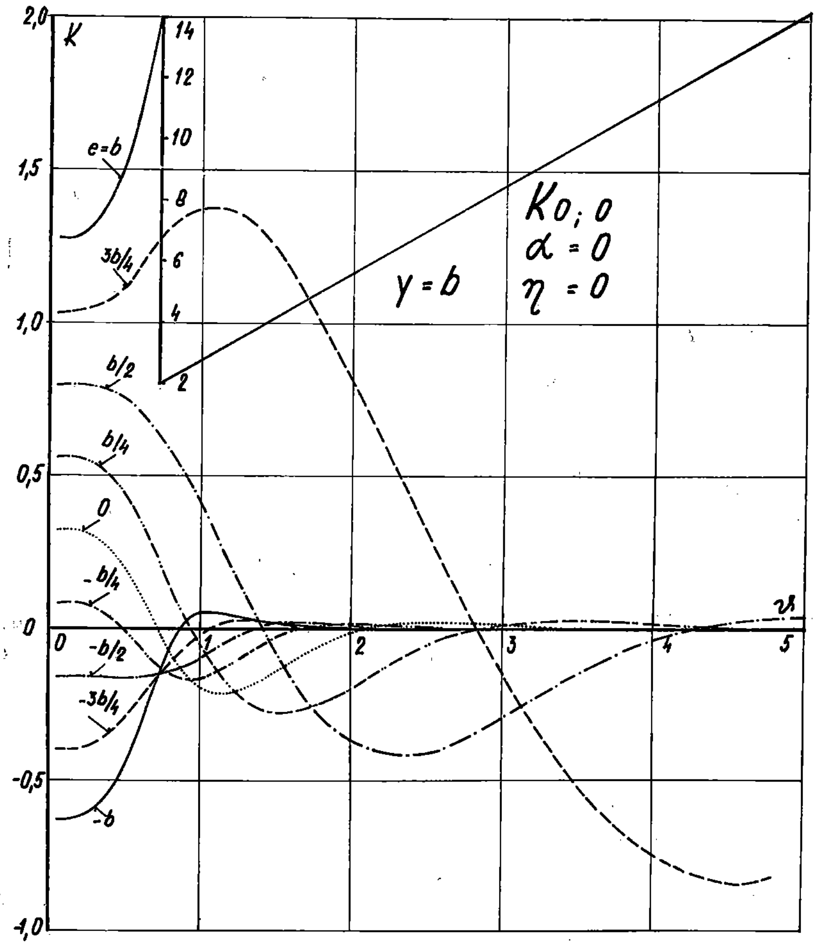


Fig. 21.

may be assumed as a mean value of a line loading considered in the foregoing section, so that

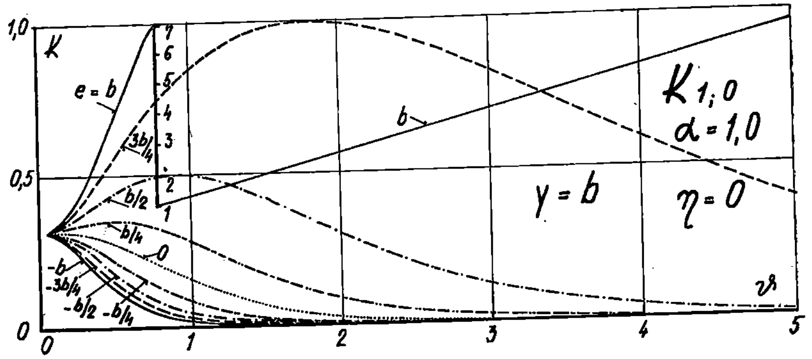
$$p^0(x) = \frac{p(x)}{2b}.$$

The constants of integration are again obtained from the boundary conditions (15) (16a) at the free edges with the deflection surface defined by Eq. (47). In this way four algebraic equations for the unknown constants A_m^0, B_m^0, C_m^0, and D_m^0 are arrived at:

At $y = +b$ from Eq. (15) it follows

(49a)
$$e^{mnb}\{[(\varepsilon - \eta)\, A_m^0 + \sqrt{(1 - \varepsilon^2)}\, B_m^0] \cos mtb + \\ + [(\varepsilon - \eta)\, \bar{B}_m^0 - \sqrt{(1 - \varepsilon^2)}\, A_m^0] \sin mtb\} + \\ + e^{-mnb}\{[(\varepsilon - \eta)\, C_m^0 - \sqrt{(1 - \varepsilon^2)}\, \bar{D}_m^0] \cos mtb + \\ + [(\varepsilon - \eta)\, \bar{D}_m^0 + \sqrt{(1 - \varepsilon^2)}\, C_m^0] \sin mtb\} - \eta\, \frac{l^4 p_m^0}{\varrho_T \pi^4 m^4} = 0\,.$$

Fig. 22.



At $y = -b$ from Eq. (15) similarly

(49b)
$$e^{mnb}\{[(\varepsilon - \eta)\, C_m^0 - \sqrt{(1-\varepsilon^2)}\, \bar{D}_m^0] \cos mtb - [(\varepsilon - \eta)\, \bar{D}_m^0 + \sqrt{(1-\varepsilon^2)}\, C_m^0]\} + e^{-mnb}\{[(\varepsilon - \eta)\, A_m^0 + \sqrt{(1-\varepsilon^2)}\, \bar{B}_m^0] \cos mtb - [(\varepsilon - \eta)\, \bar{B}_m^0 - \sqrt{(1-\varepsilon^2)}\, A_m^0] \sin mtb\} - \eta \frac{l^4 p_m^0}{\varrho_T \pi^4 m^4} = 0\,.$$

At $y = +b$ from Eq. (16a), we have

(50a)
$$e^{mnb}\left\{\left[(\eta - 1)\sqrt{\left(\frac{1+\varepsilon}{2}\right)} A_m^0 + (\eta + 1)\sqrt{\left(\frac{1-\varepsilon}{2}\right)} \bar{B}_m^0\right] \cos mtb + \left[(\eta - 1)\sqrt{\left(\frac{1+\varepsilon}{2}\right)} \bar{B}_m^0 - (\eta + 1)\sqrt{\left(\frac{1-\varepsilon}{2}\right)} A_m^0\right] \sin mtb\right\} - e^{-mnb}\left\{\left[(\eta - 1)\sqrt{\left(\frac{1+\varepsilon}{2}\right)} C_m^0 - (\eta + 1)\sqrt{\left(\frac{1-\varepsilon}{2}\right)} \bar{D}_m^0\right] \cos mtb + \left[(\eta - 1)\sqrt{\left(\frac{1+\varepsilon}{2}\right)} \bar{D}_m^0 + (\eta + 1)\sqrt{\left(\frac{1-\varepsilon}{2}\right)} C_m^0\right] \sin mtb\right\} = 0\,.$$

And finally at $y = -b$ from Eq. (16a) it follows that

(50b)
$$e^{mnb}\left\{\left[-(\eta - 1)\sqrt{\left(\frac{1+\varepsilon}{2}\right)} C_m^0 + (1 + \eta)\sqrt{\left(\frac{1-\varepsilon}{2}\right)} \bar{D}_m^0\right] \cos mtb + \left[(\eta - 1)\sqrt{\left(\frac{1+\varepsilon}{2}\right)} \bar{D}_m^0 + (\eta + 1)\sqrt{\left(\frac{1-\varepsilon}{2}\right)} C_m^0\right] \sin mtb\right\} + e^{-mnb}\left\{\left[(\eta - 1)\sqrt{\left(\frac{1+\varepsilon}{2}\right)} A_m^0 + (\eta + 1)\sqrt{\left(\frac{1-\varepsilon}{2}\right)} \bar{B}_m^0\right] \cos mtb + \left[-(\eta - 1)\sqrt{\left(\frac{1+\varepsilon}{2}\right)} \bar{B}_m^0 + (\eta + 1)\sqrt{\left(\frac{1-\varepsilon}{2}\right)} A_m^0\right] \sin mtb\right\} = 0\,.$$

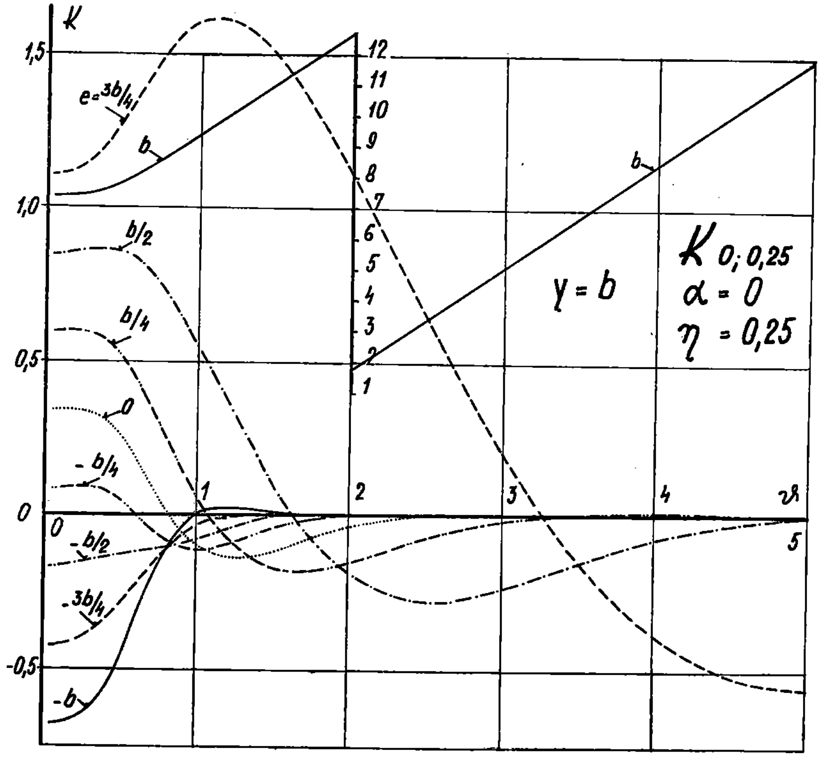


Fig. 23.

If we use the relations (40) for K, L, I, J again and further analogically to (39) and (44)

(51)
$$nb = n'\pi \qquad n' = \vartheta\sqrt{\left(\frac{1+\varepsilon}{2}\right)}$$

$$tb = t'\pi \qquad t' = \vartheta\sqrt{\left(\frac{1-\varepsilon}{2}\right)}$$

$$M_{\pi m} = e^{m\pi n'}\cos m\pi t'\ ; \quad O_{\pi m} = e^{-m\pi n'}\cos m\pi t'$$

$$N_{\pi m} = e^{m\pi n'}\sin m\pi t'\ ; \quad P_{\pi m} = e^{-m\pi n'}\sin m\pi t'\ ,$$

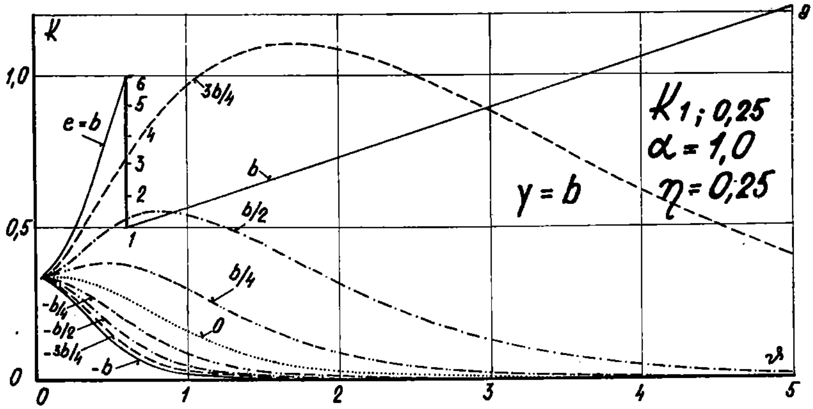
still introducing

(52)
$$C_m^{**} = \frac{l^4 p_m^0}{\varrho_T \pi^4 m^4}\eta$$

we obtain the following system of equations by adding and subtracting Eqs. (49a) and (49b), (50a) and (50b):

(53)
$$\begin{aligned}
(m_5 + m_7)(A_m^0 - C_m^0) + (m_6 + m_8)(\bar{B}_m^0 + \bar{D}_m^0) &= 0 \\
(m_5 - m_7)(A_m^0 + C_m^0) + (m_6 - m_8)(\bar{B}_m^0 - \bar{D}_m^0) &= 0 \\
(m_1 + m_3)(A_m^0 + C_m^0) + (m_2 - m_4)(\bar{B}_m^0 - \bar{D}_m^0) &= 2C_m^{**} \\
(m_1 - m_3)(A_m^0 - C_m^0) + (m_2 + m_4)(\bar{B}_m^0 + \bar{D}_m^0) &= 0\,.
\end{aligned}$$

Fig. 24.



where

(54)
$$m_1 = [M_{\pi m}(\varepsilon - \eta) - N_{\pi m}\sqrt{(1-\varepsilon^2)}]$$
$$m_2 = [M_{\pi m}\sqrt{(1-\varepsilon^2)} + N_{\pi m}(\varepsilon - \eta)]$$
$$m_3 = [O_{\pi m}(\varepsilon - \eta) + P_{\pi m}\sqrt{(1-\varepsilon^2)}]$$
$$m_4 = [-O_{\pi m}\sqrt{(1-\varepsilon^2)} + P_{\pi m}(\varepsilon - \eta)]$$
$$m_5 = \left[M_{\pi m}(\eta - 1)\sqrt{\left(\frac{1+\varepsilon}{2}\right)} - N_{\pi m}(\eta + 1)\sqrt{\left(\frac{1-\varepsilon}{2}\right)}\right]$$
$$m_6 = \left[M_{\pi m}(\eta + 1)\sqrt{\left(\frac{1-\varepsilon}{2}\right)} + N_{\pi m}(\eta - 1)\sqrt{\left(\frac{1+\varepsilon}{2}\right)}\right]$$
$$m_7 = \left[O_{\pi m}(\eta - 1)\sqrt{\left(\frac{1+\varepsilon}{2}\right)} + P_{\pi m}(\eta + 1)\sqrt{\left(\frac{1-\varepsilon}{2}\right)}\right]$$
$$m_8 = \left[O_{\pi m}(\eta + 1)\sqrt{\left(\frac{1-\varepsilon}{2}\right)} - P_{\pi m}(\eta - 1)\sqrt{\left(\frac{1+\varepsilon}{2}\right)}\right].$$

In order to simplify the writing, we put (employing Eqs. (40) and (51))

(55)
$$(m_6 - m_8) = L(\eta + 1)\sqrt{\left(\frac{1-\varepsilon}{2}\right)} + I\sqrt{\left(\frac{1+\varepsilon}{2}\right)}(\eta - 1) \equiv \sqrt{\left(\frac{1-\varepsilon}{2}\right)}n_6$$
$$(m_5 - m_7) = L(\eta - 1)\sqrt{\left(\frac{1+\varepsilon}{2}\right)} - I(\eta + 1)\sqrt{\left(\frac{1-\varepsilon}{2}\right)} \equiv -\sqrt{\left(\frac{1-\varepsilon}{2}\right)}n_5$$
$$(m_1 + m_3) = K(\varepsilon - \eta) - J\sqrt{(1-\varepsilon^2)} \equiv n_1$$
$$(m_2 - m_4) = K\sqrt{(1-\varepsilon^2)} + J(\varepsilon - \eta) \equiv \sqrt{\left(\frac{1-\varepsilon}{2}\right)}n_2$$

and further

$$n_1 n_6 + n_2 n_5 = Z$$

then the constants of integration are given by the expressions

$$(56) \qquad A_m^0 \equiv C_m^0 = \frac{C_m^{**} n_6}{Z} = C_m^{**} A_m^{0\prime} \equiv C_m^{**} C_m^{0\prime}$$

$$B_m^0 \equiv -D_m^0 = -\frac{C_m^{**} n_5}{Z} = -C_m^{**} B_m^{0\prime} \equiv C_m^{**} D_m^{0\prime}\,.$$

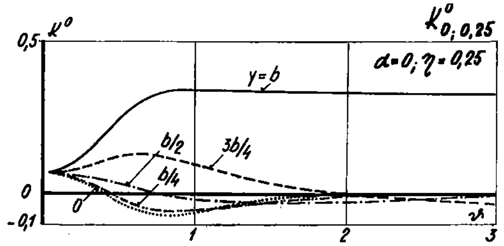


Fig. 25.

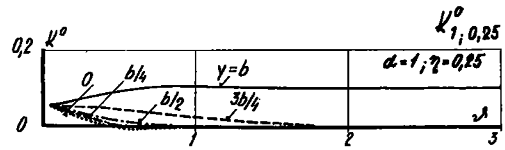


Fig. 26.

Substituting for the constants of integration into Eq. (47), we obtain the total solution of the problem of a structurally orthotropic plane structure under a loading distributed uniformly across the width and harmonical in the X-direction $p^0(x) = p_m^0 \sin m\pi x/l$. The deflection of the structure investigated is thus

$$(57) \quad w^0(x, y)_m = \frac{p_m^0 l^4}{\varrho_T \pi^4 m^4} \{1 + \eta[A_m^{0\prime}(M_{\varphi m} + O_{\varphi m}) + \bar{B}_m^{0\prime}(N_{\varphi m} - P_{\varphi m})]\} \sin \frac{m\pi x}{l}$$

or simply

$$(57a) \qquad w^0(x, y)_m = \frac{p_m^0 l^4}{\varrho_T \pi^4 m^4} [1 + K^0(y)_m] \sin \frac{m\pi x}{l}\,,$$

where the first member has the dimension of the deflection (length unity) and $K^0(y)_m$ *denotes further dimensionless coefficient* depending upon φ, ϑ, α and η

$$(58) \qquad K^0(y)_m = \eta[A_m^{0\prime}(M_{\pi m} + O_{\pi m}) + \bar{B}_m^{0\prime}(N_{\pi m} - P_{\pi m})]\,.$$

Since the m-th member for the loading $p^0(x)_m = p_m^0 \sin m\pi x/l$ equals again to the first member upon the same structure having the flexural rigidity parameter m-times greater, it is also sufficient to tabulate the coefficients $K^0(y)_1$ (for $m = 1$) only. Plots of the function K° versus ϑ are given in Figs. 25 and 26 for $\alpha = 0$, $\alpha = 1$ and $\eta = 0{\cdot}25$, and for various values of φ respectively.

4.3.3. *Procedure for a general loading*

Any general loading may be expanded into a Fourier series; for the coordinate system employed (Fig. 1) it is convenient to use the antisymmetric (odd) form with respect to $x = 0$, which is the sine-series for which $p(-x) = -p(x)$. The pertinent formulae can be found e.g. in [4]. With the aid of those formulae it is possible to solve any structurally orthotropic structure of the simple bridge type under arbitrary loading. The solution obtained in this manner may be regarded as an "exact" solution since it takes account of the actual distribution of the loading, while the solution based on the assumption of one sine half-wave loading seems to be a rather approximate one.

4.3.4. *Properties of deflection functions*

Applying the Maxwell's reciprocal theorem to the first case of the loading (line loading in the X-direction) according to which the deflection at the point y of the cross-section due to a unit loading at the point e is equal to the deflection produced at e by a unit loading at y, that is

(59)
$$w_{y,e} = w_{e,y}\,.$$

The mean deflection $(1/2b)\int_{-b}^{+b} w(x, y)\,\mathrm{d}y$ at a certain cross-section y of a system under a harmonic line loading $p(x)$ acting at e_i must be equal to the deflection $w^0(x, y)$ at $y = e_i$ due to a uniformly distributed (across the width) loading $p^0(x) = p(x)/2b$, so that

(60)
$$\frac{1}{2b}\int_{-b}^{+b} w_{e_i}(x, y)\,\mathrm{d}y = w^0(x, e_i)$$

where the deflections $w_{e_i}(x, y)$ and $w^0(x, e_i)$ are given by Eqs. (36a) and (57), respectively. Dividing Eq. (60) by the deflection $w^0(x, e_i)$, we obtain

(61)
$$\frac{1}{2bw^0(x, e_i)}\int_{-b}^{+b} w_{e_i}(x, y)\,\mathrm{d}y = 1\,.$$

Since the exact calculation of the integral in Eq. (61) is difficult, it is convenient to employ some of the methods of numerical integration. Then all the well-known general principles for numerical quadrature hold true: the step chosen must be considerably smaller than is the distance between two adjacent zero values of the function and its derivative, respectively, and the shape of the integrated function (especially as concerns the location and the number of zero points) should be similar to that of the approximating function. The most widely used methods of numerical integration are the trapezoidal and the Simpson's rules. Subdividing the width $2b$

of the structure into eight equal parts of the length $2b/8$, we have an odd number of coordinates (i.e. 9) and the integral may be expressed as follows:

(62a)
$$\int_{-b}^{+b} w_{e_i}(x, y)\,dy \cong$$
$$\cong \frac{2b}{24}\left[w_0 + w_8 + 4(w_1 + w_3 + w_5 + w_7) + 2(w_2 + w_4 + w_6)\right]_{e_i(x)}$$

according to the Simpson's rule, and

(62b)
$$\int_{-b}^{+b} w_{e_i}(x, y)\,dy \cong$$
$$\cong \frac{2b}{8}\left[\frac{w_0}{2} + w_1 + w_2 + w_3 + w_4 + w_5 + w_6 + w_7 + \frac{w_8}{2}\right]_{e_i(x)}$$

according to the trapezoidal formula.

Making use of Eq. (61), we may write

(63a)
$$\tfrac{1}{3}\left[w_0 + w_8 + 4\sum_{n=1,3,5,7} w_n + 2\sum_{m=2,4,6} w_m\right]_{e_i(x)} \cong 8w^0(x, e_i)$$

(63b)
$$\left[\tfrac{1}{2}(w_0 + w_8) + \sum_{n=1}^{7} w_n\right]_{e_i(x)} = 8w^0(x, e_i)\,.$$

Naturally, the Simpson's formula cannot be applied – with the step chosen – to areas corresponding to great ϑ. In this case the main part of the area is concentrated in the neighbourhood of the point of application of the loading, inside a strip of the width $2b/8$. Then, either the trapezoidal formula, a finer subdivision or a more exact method must be employed.

All the relations derived in the present section for $w(x, y)$ hold true even for the dimensionless coefficient $K(y)$ since it represents a reduced value of the deflection only. Especially, we have

(59a)
$$K_{e,y} = K_{y,e}$$

(61a)
$$\frac{\pi}{2b(1 + K^0_{(e_i)})}\int_{-b}^{+b} K_{e_i}(y) = 1\,.$$

The relations (59) and (61) provide an excellent check upon the numerical calculations.

4.3.5. *Special cases*

For $\alpha = 0$ and $\eta = 0$ (and consequently also $\varepsilon = 0$) all the relations pertaining to $w(x, y)_m$ and $w^0(x, y)_m$, respectively, simplify considerably, and the results coincide exactly with the results given by Guyon [6] for a non-torsion grid or derived by Massonnet [11, 13] on the basis of the analogy with the behaviour of a beam system resting on an elastic foundation as described by Hétényi [8]. In the case $\alpha = 1$, $\eta = 0$ the results of the present analysis are identical to the results of Guyon [7] derived for an isotropic plate. If $\vartheta \to 0$, $\eta = 0$ and simultaneously $\alpha = \varepsilon = 0$, the results coincide with the Engesser's solution. The case $\vartheta = 0$, $\eta = 0$ and $\alpha = \varepsilon > 0$ has been investigated in [4]; such a system deforms into a cylindrical surface even in the case that the loading is excentric with respect to the X-axis. For $\vartheta \to 0$ the results obtained by the present method are in agreement with this fact.

METHOD OF DIMENSIONLESS COEFFICIENTS FOR ANALYSIS OF STRUCTUALLY ORTHOTROPIC PLANE STRUCTURES

PART 1

The method of analysis by means of dimensionless coefficients of structurally orthotropic plane structures of simple bridge type has been derived on the basis of the analogy with material orthotropy of plates. This method takes account not only of flexural and torsional rigidities but also of the contraction ability of the structure. Any structure of the type described is defined by its outer shape and its loading, and by the three dimensionless parameters. Basic relations pertaining to the harmonic line loading and harmonic uniformly distributed (across the width) loading are derived, and the properties are discussed of deflection functions composed of a simple dimensional part and of a complicated dimensionless coefficient. The value of dimensionless coefficients are given in graphical form for limit values of the torsional parameter α and the parametr of contraction ability η in dependence on the parameter of lateral stiffness ϑ varying in the limits $(0; 5.0\rangle$.

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